From Dirac operator to supermanifolds and supersymmetries
PAI DYGEST Meeting

Jean-Philippe Michel

Département de Mathématiques,
Université de Liège

First aim

Spinor differential operators $\longrightarrow$ symplectic supermanifold

[partially with M. Grützmann and P. Xu]

Smooth manifold M = Continuous manifold + $\mathcal{C}^\infty(M) \overset{\text{loc.}}{\cong} \mathcal{C}^\infty(\mathbb{R}^n)$.

Supermanifold $\mathcal{X} = (M, \mathcal{O})$ = Cont. manifold M + $\mathcal{O}(\mathcal{X}) \overset{\text{loc.}}{\cong} \mathcal{C}^\infty(\mathbb{R}^n) \otimes \Lambda\mathbb{R}^m$.

$\mathbb{N}$-graded manifold $\mathcal{X} = (M, \mathcal{O})$ = Cont. manifold M + $\mathcal{O}(\mathcal{X}) \overset{\text{loc.}}{\cong} \mathcal{C}^\infty(\mathbb{R}^n) \otimes \mathcal{S}V$, with V a $\mathbb{N}^*$-graded vector space.

Example: $\mathcal{X} = \Pi TM$, $T[1]M$, then $\mathcal{O}(\mathcal{X}) = \Omega(M)$.

Introduction to quantization

	Classical	Quantum
Phase space	$(\mathcal{M}, \omega)$ symplectic manifold	$\mathcal{H}$ Hilbert space
Observables	$\mathcal{P} \subset \mathcal{C}^\infty(\mathcal{M})$ Poisson algebra	$\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ Associative algebra
Symmetries	$G \subset \mathrm{Sympl}(\mathcal{M}, \omega)$	$G \subset \mathrm{U}(\mathcal{H})$

- Symbolic calculus for pseudo-differential operators [Hörmander, Unterberger,...],
- Geometric quantization [Kostant, Souriau,...],
- Orbit method [Kirillov, Kostant, Duflo, Dixmier, ...],
- Deformation of Poisson algebras [Gerstenhaber, Lichnerowicz, Flato, Fedosov, Kontsevich, ...].

Examples of quantization

	Classical	Quantum
Phase space	$\mathfrak{g}^*$	$\mathrm{L}^2(G)$
Observables	$\mathcal{S}\mathfrak{g}$	$\mathfrak{U}(\mathfrak{g})$
	graded algebra	filtered algebra
	commutative	non-commutative

Filtered algebra:
$\mathcal{A} = \bigcup_{k\in\mathbb{N}} \mathcal{A}_k$, $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots$, and $\mathcal{A}_k \cdot \mathcal{A}_l \subset \mathcal{A}_{k+l}$.

Associated graded algebra:
$\mathrm{gr}\,\mathcal{A} := \bigoplus_{k\in\mathbb{N}} \mathcal{A}_k/\mathcal{A}_{k-1}$.
If $[\mathcal{A}_k, \mathcal{A}_l] \subset \mathcal{A}_{k+l-1}$, then $\mathrm{gr}\,\mathcal{A}$ is a commutative Poisson algebra.

Examples of quantization

	Classical	Quantum
Phase space	(V,ω) | $(V[1],\mathrm{g})$	$\mathrm{L}^2(L_\omega)$ | ΛL_{g}
Observables	$\mathcal{S}V$ | $\Lambda_{\mathbb{C}}V$	$\mathcal{W}(V)$ | $\mathbb{C}\mathrm{l}(V)$
	graded algebra	filtered algebra
	commutative	non-commutative

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Examples of quantization

	Classical	Quantum
Phase space	T^*M	$\mathrm{L}^2(M)$
Observables	$\mathrm{Pol}(T^*M)$	$\mathcal{D}(M)$
	graded algebra	filtered algebra
	commutative	non-commutative

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Problematic

For spin 0 free particles on a pseudo-Riemannian manifold (M, g):

	Classical	Quantum
Phase space	T^*M |	$\mathrm{L}^2(M)$ |
Observables	$\mathrm{Pol}(T^*M)$ |	$\mathcal{D}(M)$ |
Equations of motion	$\dot{P} = \{H, P\}$	$\dot{\widehat{P}} = \frac{\mathrm{i}}{\hbar}[\widehat{H}, \widehat{P}]$
$(H = p^2)$	$\nabla_{\dot{x}}\dot{x} = 0$ |	$-\Delta f = m^2 f$ |

Problematic

For spin 0 and spin $\frac{1}{2}$ free particles on a pseudo-Riemannian manifold (M, g):

	Classical	Quantum
Phase space	T^*M | ?	$\mathrm{L}^2(M)$ | $\mathrm{L}^2(M,S)$
Observables	$\mathrm{Pol}(T^*M)$ | ?	$\mathcal{D}(M)$ | $\mathcal{D}(M,S)$
Equations of motion $(H = p^2)$	$\dot{P} = \{H, P\}$ $\nabla_{\dot{x}}\dot{x} = 0$ | ?	$\dot{\widehat{P}} = \frac{\mathrm{i}}{\hbar}[\widehat{H}, \widehat{P}]$ $-\Delta f = m^2 f$ | $\mathrm{i}D\psi = m\psi$

Spin bundle S:
$\mathrm{End}S = \mathbb{C}\mathrm{l}(M)$ if (M, g) spin manifold of even dimension.

$$\gamma : \Omega^{\mathbb{C}}(M) \to \Gamma(\mathbb{C}\mathrm{l}(M))$$

Dirac operator:

$$D := \gamma(dx^i)\nabla_i.$$

A solution

Is there a filtration on $\mathcal{D}(M, S)$ such that $\mathrm{gr}\,\mathcal{D}(M, S)$ is a Poisson algebra?

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Is there a filtration on $\mathcal{D}(M,S)$ such that $\mathrm{gr}\,\mathcal{D}(M,S)$ is a Poisson algebra?

Usual filtration: ∇_X order 1 and $\gamma(dx)$ order 0,

- $[\gamma(u),\gamma(v)] = 2\mathrm{g}(u,v) \Longrightarrow \mathrm{gr}\,\mathcal{D}(M,S)$ is not commutative.

Getzler's filtration: ∇_X order 1 and $\gamma(dx)$ order 1,

- leads to the index theorem for the Dirac operator [Getzler '83],
- it depends on ∇,
- $[\nabla_X,\nabla_Y] = R(X,Y) \in \Gamma(\mathbb{C}\mathrm{l}_2(M)) \Longrightarrow \mathrm{gr}\,\mathcal{D}(M,S)$ is not commutative.

A solution (continued)

Hamiltonian filtration: ∇_X order 2 and $\gamma(dx)$ order 1,

Proposition (Grützmann-M.-Xu)

$\mathrm{gr}\,\mathcal{D}(M,S) \cong \mathcal{O}(\mathbb{T}M)$, *where* $\mathbb{T}M = T^*[2]M \oplus T[1]M$ *is a symplectic graded manifold.*

$\mathbb{T}M = T^*[2]M \oplus T[1]M$ admits coordinates (x^i, ξ^i, p_i), of degree $0, 1, 2$, and potential 1-form [Rothstein '91; Roytenberg 2002]:

$$\alpha = p_i dx^i + \mathrm{g}_{ij}\xi^i d^\nabla \xi^j.$$

Equation of motion $(H = p^2)$: $\nabla_{\dot{x}}\dot{x} = R(s)\dot{x}, \quad s \in \Omega_2^{\mathbb{C}}(M),$

- Equation of Papapetrou for a spinning particle on (M, g),
- $\mathbb{T}M$: phase space of a spinning pseudo-particle on (M, g) [Berezin-Marinov '77].

Second aim

Symmetries of Laplacian $\oplus$ Dirac operator
$\longrightarrow$ Lie superalgebras of supersymmetries

[partially with J. Šilhan]

A Lie superalgebra $(\mathfrak{g}, [\cdot,\cdot])$ is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ endowed with a bilinear operation $[\cdot,\cdot]$ such that

- $(\mathfrak{g}_0, [\cdot,\cdot])$ is a Lie algebra,
- $[\cdot,\cdot] : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ defines a linear representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$,
- $[\cdot,\cdot] : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is symmetric and $\mathfrak{g}_0$-equivariant.

Symmetries of Laplacian

Let Δ be the pseudo-Euclidian Laplacian on $M = \mathbb{R}^{p+q}$, $n = p + q$.

Determine the algebra of symmetries of $\ker \Delta$, i.e.

$$\mathcal{A} := \{\text{diff. op.} D_1 \,|\, \exists D_2, \Delta D_1 = D_2 \Delta\}/(\Delta).$$

- Pb due to Witten, motivated by higher spin gauge theory [Vasiliev et al.].
- $\mathcal{A}_1/\mathcal{A}_0 \cong \mathfrak{o}(n+2) \cong$ Lie algebra of conformal symmetries. If $L_X \mathrm{g} = e^{\Upsilon}\mathrm{g}$, then
$$\Delta \circ (X^i \partial_i + \frac{n-2}{2n}\partial_i X^i) = (X^i \partial_i + \frac{n+2}{2n}\partial_i X^i) \circ \Delta.$$
- $\mathcal{A}_2$ determined in [Boyer-Kalnins-Miller '76].
- Via the properties of principal symbol maps, $\mathrm{gr}\,\mathcal{A} \leq \mathcal{K}$ with
$$\mathcal{K} = \{P \in \mathrm{Pol}(T^*M) |\, \{p^2, P\} \in (p^2)\}/(p^2) = \{\text{Conf. Killing tensors}\}.$$

Symmetries of Laplacian (II)

Idea: There exists a unique $\mathfrak{o}(n+2)$-equivariant quantization map
[Duval-Lecomte-Ovsienko '99]

$$\mathrm{CEQ} : \mathrm{Pol}(T^*\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n).$$

Together with: Classification $\mathfrak{o}(n+2)$-invariant diff. op. + symplectic reduction:

Theorem (Eastwood '05; M. '11)

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\mathrm{CEQ}} & \mathcal{A} \\ \| & & \| \\ \mathbb{C}[\mathcal{O}_{min}] & & \mathfrak{U}(\mathfrak{o}(n+2))/\mathcal{J} \end{array}$$

where $\mathcal{O}_{min}$ *is the min. coad. orbit of* $\mathrm{O}(p+1,q+1)$ *and* $\mathcal{J}$ *the Joseph ideal.*

Symmetries of the Dirac operator

Theorem (M. 2012)

There exist unique conformally equivariant superization and quantization maps

$$\{Tensors\} \xrightarrow{\text{CES}} \mathcal{O}(\mathcal{M}) \xrightarrow{\text{CEQ}} \mathcal{D}(M,S).$$

Moreover,

$$\{KC\ \ Tensors\ \} \xrightarrow{\text{CES}} \{Supercharges\} \xrightarrow{\text{CEQ}} \{Sym.\ker D\}.$$

Generalisation of

- Symmetries of order 1 of the Dirac operator [Benn-Kreiss 2004],
- Supercharges of order 1 [Gibbons-Rietdijk-van Holten '93].

Algebraic structure of $\mathcal{A}$:

- if $n = \dim M$ is odd, $\mathcal{A} \cong \mathfrak{U}(\mathfrak{o}(n+2))/\mathcal{J}_D$,
- if n is even, $S = S^+ \oplus S^-$ and $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, with $\mathcal{A}^0 \cong \mathfrak{U}(\mathfrak{o}(n+2))/\mathcal{J}_D$.

Symmetries of $\Delta \oplus D$

$$\begin{pmatrix} \Delta & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} f \\ \psi \end{pmatrix} = 0$$

$\iff$ E.-L. eq. of the free supersymmetric Lagrangian [Wess-Zumino '74].

Symmetries :
$$\begin{pmatrix} \Delta & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a_1 & \alpha_1^- \\ \alpha_1^+ & A_1 \end{pmatrix} = \begin{pmatrix} a_2 & \alpha_2^- \\ \alpha_2^+ & A_2 \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & D \end{pmatrix}$$

Proposition (M.-Šilhan)

In dimension 3, $\mathcal{A} = \mathfrak{U}(\mathfrak{g})/\mathcal{J}$ *where* $\mathfrak{g} = \mathfrak{spo}(4|2)$,

$$\mathfrak{g}_0 = \mathfrak{o}(5, \mathbb{C}) \oplus \mathbb{C} \quad \textit{and} \quad \mathfrak{g}_1 = \{\textit{Twistor-Spinors}\}.$$

Geometric realization: in dimension 3, $\mathfrak{spo}(4|2) \hookrightarrow \mathrm{Vect}(\Pi S^*)$ and

$$\text{symmetries } \Delta \oplus D \iff \text{symmetries } \Box \in \mathcal{D}(\Pi S^*),$$

where $\Box = \epsilon\Delta \oplus D \oplus \epsilon^*$ and ϵ^* is the spinor metric.